Note that the controlled subsystem of four or five equations in this problem is an uncontrolled scalar control, even if it depends on all the phase variables of the problem.

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ON THE CONSTRUCTION OF A BOUNDED CONTROL IN OSCILLATORY SYSTEMS*

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The motion of a linear controlled system from any initial state to a given final state is considered when there are geometric constraints on the control. One way of constructing the control when there are no constraints is to use a control signal formed by a linear combination of natural motions of the uncontrolled system /1, 2/. In the present paper this control method is used when there are geometrical constraints on the control functions. Sufficient conditions are obtained, under which this control law solves the problem in finite time. The same approach is applied to a multifrequency system of linear oscillators (pendulums) which are scalarly controlled. The control law is obtained and the process time is estimated. The control is also found for a two-mass system which contains an oscillatory unit.

1. Formulation of the problem. Consider a linear controlled dynamic sytstem with a bounded control

$$x^{\cdot} = A(t)x + B(t)u \tag{1.1}$$

$$|u(t)| \leqslant a, \quad a > 0 \tag{1.2}$$

Here, x is the n-dimensional vector of phase coordinates, u is the m-dimensional control vector, $A(t)$ and $B(t)$ are $n \times n$ and $n \times m$ matrices respectively, piecewise continuously dependent on time t, and a is a positive constant.

We shall construct the control $u(t)$ which satisfies the constraint (1.2) for $t \in [t_0, T]$ and moves the system from the initial state

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$$x(t_0) = x^0 \tag{1.3}$$

to the final state

$$x(T) = x^1 \tag{1.4}$$

Here, x^0, x^1 are any given n-dimensional vectors, the initial instant t_0 is assumed to be given, and the instant T when the process terminates may be either fixed or free $(T > t_0)$.

The fundamental matrix $\Phi(t)$ of the homogeneous system (1.1) is given by the conditions

$$\Phi^{\cdot} = A(t)\Phi, \quad \Phi(t_0) = E_n \tag{1.5}$$

Here E_n is the $n \times n$ identity matrix. We write the solution of system (1.1) which satisfies the initial condition (1.3)

$$x(t) = \Phi(t)\left[x^0 + \int_{t_0}^{t} \Phi^{-1}(\tau) B(\tau) u(\tau)\, d\tau\right] \tag{1.6}$$

Substituting the boundary condition (1.4) into (1.6), we obtain

$$\int_{t_0}^{T} \Phi^{-1}(t) B(t) u(t)\, dt = x^*, \quad x^* = \Phi^{-1}(T) x^1 - x^0 \tag{1.7}$$

The required control $u(t)$ must therefore satisfy the constraint (1.2) and condition (1.7).

Recall that construction of the time-optimal control of system (1.1), (1.2), which moves it from state (1.3) to state (1.4) in the shortest time, reduces, by the maximum principle /3/, to solving a system of n transcendental equations. The method given below, which was proposed in /1, 2/ for the case when there are no constraints on the control, does not provide time-optimality, but is simpler for calculation and realization.

2. The control method. We seek the control that solves problem (1.1)-(1.4) in the form

$$u(t) = Q^T(t)c, \quad Q(t) = \Phi^{-1}(t) B(t) \tag{2.1}$$

Here, c is an n-dimensional constant vector, $Q(t)$ is an $n \times m$ matrix, and T denotes transposition. Substituting (2.1) into (1.7), we obtain

$$R(T)c = x^*, \quad R(T) = \int_{t_0}^{T} Q(t) Q^T(t)\, dt \tag{2.2}$$

($R(T)$ is a symmetric $n \times n$ matrix).

We take the quadratic form (v is an n-dimensional constant vector)

$$(R(T)v, v) = \int_{t_0}^{T} |Q^T(t) v|^2\, dt \geqslant 0 \tag{2.3}$$

It follows from (2.3) that $R(T)$ is a non-negative definite matrix. We know /4/ that, when system (1.1) is completely controllable, the integral (2.3) does not vanish for any constant $v \neq 0$. Then, $R(T)$ is positive definite and the linear system of algebraic Eqs.(2.2) has the unique solution

$$c = R^{-1}(T) x^* \tag{2.4}$$

We will give simple sufficient conditions for the constraint (1.2) to be satisfied for the control law (2.1).

Theorem. For some $T > t_0$ let the matrix $R(T)$ be non-singular, i.e., the condition for complete controllability holds, and for any n-dimensional vector v, let us have the inequalities

$$|Q^T(t) K(T) v| \leqslant \mu(T) |v|, \quad t \in [t_0, T] \tag{2.5}$$

$$|R(T) K(T) v| \geqslant \lambda(T) |v| \tag{2.6}$$

Here, $K(T)$ is a non-singular $n \times n$ matrix, $\mu(T) > 0$ and $\lambda(T) > 0$ are positive scalars, and v is a constant n-dimensional vector, while the inequality (2.5) must hold for all $t \in [t_0, T]$.

Then, if we have the condition

$$|x^*| \leqslant a\lambda(T) \mu^{-1}(T) \tag{2.7}$$

the control $u(t)$ given by (2.1), (2.4) takes system (1.1) from state (1.3) to state (1.4) at the instant T and satisfies the constraint (1.2) for all $t \in [t_0, T]$.

Proof. The control (2.1), (2.4) is constructed in such a way that conditions (1.3) and (1.4) hold. By (2.1) and (2.4), we have

$$|u(t)| = |Q^T(t)R^{-1}(T)x^*| = |Q^T(t)K(T)K^{-1}(T)R^{-1}(T)x^*|$$

We use inequality (2.5)

$$|u(t)| \leqslant \mu(T)|K^{-1}(T)R^{-1}(T)x^*|$$

In the inequality obtained we put $x^* = R(T)K(T)v$ and first apply (2.6) and then (2.7):

$$|u(t)| \leqslant \mu(T)|v| \leqslant \mu(T)\lambda^{-1}(T)|R(T)K(T)v| = \mu(T)\lambda^{-1}(T)|x^*| \leqslant a$$

We have thus shown that the constraint (1.2) holds, which proves the theorem.

Notes. 1°. The non-singular matrix $K(T)$ in (2.5) and (2.6) can be chosen arbitrarily, and in particular, we can take $K = E_n$. The arbitrary choice of $K(T)$ can be useful, since it extends the range in which our sufficient conditions are applicable.

2°. In the case of the identity matrix $K = E_n$, the number $\lambda(T)$ is, by (2.6), a lower bound for the least eigenvalue of the matrix $R(T)$.

3°. To calculate the control (2.1), we have to solve the linear system of Eqs.(2.2), whereas in the time-optimal case we have to solve a system of transcendental equations.

4°. The control (2.1) is a continuous function of time, whereas the time-optimal control is in general discontinuous.

3. A system of controlled oscillators. Consider a system of harmonic oscillators, controlled scalarly:

$$\xi_i'' + \omega_i^2\xi_i = u \tag{3.1}$$

Here, ξ_i are generalized coordinates, the constants $\omega_i > 0$ are the natural frequencies of the oscillators, and throughout $i = 1, \ldots, n$; u is the scalar control, on which is imposed the constraint (a is a constant)

$$|u(t)| \leqslant a \tag{3.2}$$

As a mechanical model of system (3.1) we can take a system of mathematical pendulums, suspended from a support G which moves horizontally with acceleration u (Fig.1). The ξ_i, equal to $l_i\varphi_i$, are the small linear deviations of the pendulums from their points of suspension, where l_i is the length, and φ_i the angle of deviation of the pendulum from the vertical.

Another mechanical model of system (3.1) is a set of masses connected by springs to the support G. The system as a whole moves translationally and horizontally, ξ_i being the spring elongation, and u the acceleration of the body G (Fig.2).

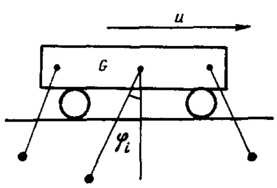

Fig.1

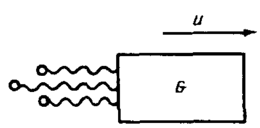

Fig.2

Let us find the control $u(t)$ which satifies the constraint (3.2) and moves the system (3.1) from its intial state at $t_0 = 0$:

$$\xi_i(0) = \xi_i^0, \quad \xi_i'(0) = \eta_i^0 \tag{3.3}$$

to the given final state

$$\xi_i(T) = \xi_i^1, \quad \xi_i'(T) = \eta_i^1 \tag{3.4}$$

We shall assume that the frequencies ω_i are positive and distinct. There is no loss of generality if we number them in increasing order, put $\omega_0 = 0$, and introduce the notation

$$\Omega = \min_{0\leqslant k\leqslant n-1}(\omega_{k+1} - \omega_k) > 0 \tag{3.5}$$

$$0 = \omega_0 < \omega_1 < \ldots < \omega_n$$

Note that, when $\Omega > 0$, system (3.1) is completely controllable /5/. If some frequencies are the same, the system becomes uncontrollable. For, if the initial states of two oscillators with equal frequencies are different, there is no control by which we can arrange for simultaneous extinction of the oscillations of these two oscillators: the phase difference of their oscillations will remain constant.

Using the change of variables

$$\xi_i^{\cdot} = y_i, \quad \xi_i = \omega_i^{-1} z_i \tag{3.6}$$

we can reduce system (3.1) to the form

$$y_i^{\cdot} = -\omega_i z_i + u, \quad z_i^{\cdot} = \omega_i y_i \tag{3.7}$$

The phase vector of system (3.7) is a $2n$-dimensional column vector, formed from the components of vectors y and z.

It can be shown that the fundamental matrix of the homogeneous system (3.7), given by conditions (1.5), is orthogonal and has the form

$$\Phi(t) = \begin{Vmatrix} \operatorname{diag}(\cos \omega_i t) & \operatorname{diag}(-\sin \omega_i t) \\ \operatorname{diag}(\sin \omega_i t) & \operatorname{diag}(\cos \omega_i t) \end{Vmatrix} \tag{3.8}$$

$$\Phi^{-1}(t) = \Phi^T(t)$$

Here, $\operatorname{diag}(a_i)$ denotes an $n \times n$ diagonal matrix with diagonal elements a_i.

For system (3.7), matrices $B(t)$ and $Q(t)$ are $2n$-dimensional column vectors. By (3.7, (2.1) and (3.8), their elements are

$$B_i = 1, \quad B_{n+i} = 0, \quad Q_i(t) = \cos \omega_i t, \quad Q_{n+i} = -\sin \omega_i t \tag{3.9}$$

From (3.9) and the second of Eqs.(2.2), we have

$$QQ^T = \begin{Vmatrix} Q^1 & Q^0 \\ Q^0 & Q^2 \end{Vmatrix}, \quad R(T) = \begin{Vmatrix} R^1 & R^0 \\ R^0 & R^2 \end{Vmatrix} \tag{3.10}$$

$$R^k = \int_0^T Q^k \, dt, \quad k = 0, 1, 2$$

Here, Q^k, R^k are $n \times n$ matrices. Their elements are calculated by means of (3.9) and (3.10) (throughout, $i, j = 1, \ldots, n$)

$$Q_{ij}^1 = \cos \omega_i t \cos \omega_j t, \quad Q_{ij}^2 = \sin \omega_i t \sin \omega_j t \tag{3.11}$$

$$Q_{ij}^0 = -\cos \omega_i t \sin \omega_j t; \quad R_{ii}^{1,2} = \frac{T}{2} \pm \frac{\sin 2\omega_i T}{4\omega_i}$$

$$R_{ij}^{1,2} = \frac{\sin(\omega_i - \omega_j) T}{2(\omega_i - \omega_j)} \pm \frac{\sin(\omega_i + \omega_j) T}{2(\omega_i + \omega_j)}$$

$$R_{ii}^0 = \frac{\cos 2\omega_i T - 1}{4\omega_i}, \quad R_{ij}^0 = \frac{\cos(\omega_i - \omega_j) T - 1}{2(\omega_i - \omega_j)} + \frac{\cos(\omega_i + \omega_j) T - 1}{2(\omega_i + \omega_j)}, \quad i \neq j$$

Note that, by condition (3.5),

$$\omega_i \geqslant \Omega, \quad |\omega_i - \omega_j| \geqslant \Omega, \quad \omega_i + \omega_j \geqslant 3\Omega, \quad i \neq j \tag{3.12}$$

Using (3.12), we can obtain estimates for the elements (3.11) of the matrix $R(T)$:

$$|R_{ii}^k - {}^1/_2 T| \leqslant {}^1/_4 \Omega^{-1}, \quad |R_{ii}^0| \leqslant {}^1/_2 \Omega^{-1} \tag{3.13}$$

$$|R_{ij}^k| \leqslant {}^1/_2 |\omega_i - \omega_j|^{-1} + {}^1/_2 (\omega_i + \omega_j)^{-1} \leqslant {}^2/_3 \Omega^{-1}$$

$$|R_{ij}^0| \leqslant {}^4/_3 \Omega^{-1}; \quad k = 1, 2; \quad i \neq j$$

Under conditions (2.5) and (2.6) we put $K(T) = E_{2n}$ and find $\mu(T)$ and $\lambda(T)$. We estimate the left-hand side of inequality (2.5) by using the Cauchy inequality and expressions (3.9) for the components of the vector $Q(t)$:

$$|Q^T(t) v| \leqslant |Q(t)| |v| = n^{1/2} |v|$$

Consequently, in (2.5) we can put

$$\mu(T) = n^{1/2} \tag{3.14}$$

We will estimate the left-hand side of inequality (2.6). For any vector v, we have

$$|R(T) v| = |{}^1/_2 T v + [R(T) - {}^1/_2 T E_{2n}] v| \geqslant {}^1/_2 T |v| - |Mv|, \tag{3.15}$$

$$M = R(T) - {}^1/_2 T E_{2n}$$

Here we introduce the symmetric $2n \times 2n$ matrix M. For its elements, using relations (3.10) and (3.13) for the matrix $R(T)$, we obtain the estimates

$$
\begin{aligned}
&|M_{ii}| \leqslant {}^1/_4\Omega^{-1}, \quad |M_{n+i,n+i}| \leqslant {}^1/_4\Omega^{-1} \\
&|M_{ij}| \leqslant {}^2/_3\Omega^{-1}, \quad |M_{n+i,n+j}| \leqslant {}^2/_3\Omega^{-1} \\
&|M_{i,n+i}| \leqslant {}^1/_2\Omega^{-1}, \quad |M_{i,n+j}| \leqslant {}^4/_3\Omega^{-1}, \quad i \neq j
\end{aligned}
\tag{3.16}
$$

By the Cauchy inequality, we have (the summation here and in (3.18) is from 1 to $2n$)

$$
|Mv|^2 = \sum_i \Big(\sum_j M_{ij}v_j\Big)^2 \leqslant \sum_i \Big[\Big(\sum_i M_{ij}^2\Big)\Big(\sum_j v_j^2\Big)\Big] = |v|^2 \sum_{i,j} M_{ij}^2 \tag{3.17}
$$

Recalling estimates (3.16) and that M is symmetric, we obtain

$$
\sum_{i,j} M_{ij}^2 \leqslant \frac{2n}{(4\Omega)^2} + \frac{2(n^2-n)4}{(3\Omega)^2} + \frac{2n}{(2\Omega)^2} + \frac{2(n^2-n)16}{(3\Omega)^2} = \frac{5n(64n-55)}{72\Omega^2} \tag{3.18}
$$

From (3.17) and (3.18) we have

$$
\begin{aligned}
&|Mv| \leqslant k_n\Omega^{-1}v \\
&k_n = [5n(64n-55)/72]^{1/2}, \quad n \geqslant 1
\end{aligned}
\tag{3.19}
$$

Using (3.15) and (3.19), we obtain

$$
|R(T)v| \geqslant ({}^1/_2T - k_n\Omega^{-1})|v| \tag{3.20}
$$

Thus, condition (2.6) holds if $T > 2k_n\Omega^{-1}$. Then, comparing (2.6) and (3.20), we obtain

$$
\lambda(T) = {}^1/_2T - k_n\Omega^{-1} > 0 \tag{3.21}
$$

Substituting relations (3.14), (3.21) in (2.7) and solving for T, we get

$$
T \geqslant 2n^{1/2}a^{-1}|x^*| + 2k_n\Omega^{-1} \tag{3.22}
$$

The vector x^* is given by the second relation of (1.7), while vectors x^0, x^1 are, by (3.6), (3.3) and (3.4):

$$
\begin{aligned}
x^0 &= \{y_i(0),\ z_i(0)\}^T = \{\eta_i^0,\ \omega_i\xi_i^0\}^T \\
x^1 &= \{y_i(T),\ z_i(T)\}^T = \{\eta_i^0,\ \omega_i\xi_i^1\}^T
\end{aligned}
\tag{3.23}
$$

In the control law (2.1) we substitute the elements of $Q(t)$ from (3.9):

$$
u(t) = \sum_{i=1}^{n} (c_i \cos\omega_i t - c_{n+i}\sin\omega_i t) \tag{3.24}
$$

By the theorem of Par.2, under condition (3.22), the control (3.24) in which the vector c is given by (2.4), and the matrix $R(T)$ is given by (3.10), (3.11), satisfies the constraint (3.2) and moves system (3.7) (or (3.1)) from the initial state (3.3) to the final state (3.4) in time T. Note that the time T increases as $|x^*|$ increases, as the scope (i.e., a) of the control falls, and as the natural frequencies come closer together, i.e., as Ω falls.

4. A special case. Consider the problem of extinguishing the initial oscillations, i.e., the problem of taking the system to its equilibrium state. In this case we have $x^1 = 0$, and from the second of Eqs.(1.7) and (3.23), we obtain ($E(t)$ is the energy of the oscillation)

$$
|x^*|^2 = \sum_{i=1}^{n} [(\eta_i^0)^2 + \omega_i^2(\xi_i^0)^2] = 2E_0, \quad E_0 = E(0) \tag{4.1}
$$

$$
E(t) = \frac{1}{2}\sum_{i=1}^{n} \{[\xi_i^{\cdot}(t)]^2 + \omega_i^2[\xi_i(t)]^2\} \tag{4.2}
$$

Using (4.1), we can rewrite (3.22) as

$$
T \geqslant 2a^{-1}(2nE_0)^{1/2} + 2k_n\Omega^{-1} \tag{4.3}
$$

Under condition (4.3), the control (3.24) moves system (3.1) from the initial state (3.3) to the equilibrium state $\xi_i = \xi_i^{\cdot} = 0$.

In the special case $n = 1$ the minimum time which satisfies condition (4.3) is equal to (we use the second relation of (3.19))

$$
T^* = 2a^{-1}(2E_0)^{1/2} + ({}^5/_2)^{1/2}\omega_1^{-1} \tag{4.4}
$$

We compare the time (4.4) with the optimal control time under the condition

$$
\varepsilon = aE_0^{-1/2}\omega_1^{-1} \ll 1 \tag{4.5}
$$

which signifies that the control is relatively small. Then, the approximate optimal control of system (3.1), (3.2) with $n=1$, which is constructed in /5/ by the small parameter method /6/, is

$$u = -a \operatorname{sign}(\xi_1^{\cdot}) \tag{4.6}$$

while the phase coordinates are

$$\begin{aligned} \xi_1 &= \pm (2E)^{1/2}\omega_1^{-1}\cos(\omega_1 t + \alpha) \\ \xi_1^{\cdot} &= \mp (2E)^{1/2}\sin(\omega_1 t + \alpha) \end{aligned} \tag{4.7}$$

Here, the energy E and phase α are slow variables.

We differentiate the energy E of (4.2) with respect to t, and use Eqs.(3.1), (4.6) and (4.7):

$$E^{\cdot} = \xi_1^{\cdot}(\xi_1^{\cdot\cdot} + \omega_1^2\xi_1) = -\xi_1^{\cdot}u = -a\,|\,\xi_1^{\cdot}\,| = -a\,(2E)^{1/2}\,|\sin(\omega_1 t + \alpha)\,|$$

In accordance with the averaging method of /6/, we average the right-hand side of this last equation with respect to t, regarding E and α as constants. We obtain the equation of the first approximation, which we integrate:

$$[2E(t)]^{1/2} = (2E_0)^{1/2} - 2\pi^{-1}a\omega_1 t$$

Hence it follows that the time T^0, needed for extinction of the oscillations $(E(T^0) = 0)$, is equal to

$$T^0 = {}^1/_2\pi a^{-1}(2E_0)^{1/2} \tag{4.8}$$

Expressions (4.4) and (4.8) have to be compared under condition (4.5), under which the approximate expression (4.8) is obtained. The second term in (4.4) is here much smaller than the first, while the principal parts of (4.4) and (4.8) differ by a factor. We have

$$T^*/T^0 \approx 4/\pi = 1.273 \quad (\varepsilon \ll 1)$$

This relation gives an estimate of the closeness of the results obtained by the present control method and the time-optimal control.

5. A pendulum with a controlled point of suspension. We again consider the systems shown in Figs.1 and 2, in the case of a single oscillator $(n = 1)$ but assuming a displacement ξ_0 of the support G. The equations of motion and the constraint (3.2) take the form

$$\xi_1^{\cdot\cdot} + \omega_1^2\xi_1 = u, \quad \xi_0^{\cdot\cdot} = u, \quad |\,u\,| \leqslant a \tag{5.1}$$

All the notation here is the same as in Par.3. Note that the displacements ξ_0 and ξ_1 are measured in opposite directions, so that the absolute oscillator displacement is $\xi_0 - \xi_1$.

We also take a modified statement of the problem, in which the systems of Figs.1 and 2 are controlled, not by accelerating the support G, but by means of a force F applied to the body G and bounded in value by the constant F_0. Then, instead of relations (5.1), we have the equations and the constraint

$$\xi_1^{\cdot\cdot} + \omega_1^2\xi_1 = \xi_0^{\cdot\cdot}, \quad (m_0 + m_1)\xi_0^{\cdot\cdot} - m_1\xi_1^{\cdot\cdot} = F, \quad |\,F\,| \leqslant F_0 \tag{5.2}$$

where m_0 is the mass of the support G, and m_1 is the mass of the oscillator. We introduce the system centre of mass coordinate

$$\xi = [(m_0 + m_1)\xi_0 - m_1\xi_1](m_0 + m_1)^{-1}$$

and transform relations (5.2) to

$$\begin{aligned} \xi_1^{\cdot\cdot} + \omega_1^2(m_0 + m_1)m_0^{-1}\xi_1 &= Fm_0^{-1} \\ \xi^{\cdot\cdot} = F(m_0 + m_1)^{-1}, &\quad |\,F\,| \leqslant F_0 \end{aligned} \tag{5.3}$$

The change of variables and constants

$$\begin{aligned} &\xi' = (m_0 + m_1)m_0^{-1}\xi, \quad (\omega')^2 = \omega_1^2(m_0 + m_1)m_0^{-1} \\ &u = Fm_0^{-1} \end{aligned}$$

transform relations (5.3), apart from the notation, to the form (5.1). Thus relations (5.1) also describe systems which are controlled by means of a bounded force.

To simplify (5.1), we make the change of variables

$$\begin{aligned} &\xi_1 = \omega_1^{-2}ay, \quad \xi_0 = \omega_1^{-2}az \\ &t = \omega_1^{-1}t', \quad u = au' \end{aligned} \tag{5.4}$$

After the replacement (5.4), relations (5.1) become

$$y^{\cdot\cdot} + y = u, \quad z^{\cdot\cdot} = u, \quad |\,u\,| \leqslant 1 \tag{5.5}$$

Henceforth we shall consider the system in the form (5.5) and denote by points derivatives with respect to the new time t', while the primes of t' and u' in (5.5) will be omitted.

Let us construct the control $u(t)$ which satifies the condition $|\,u\,| \leqslant 1$ and moves system (5.5) from the given initial state

$$y(0) = y^0, \quad y^{\cdot}(0) = v^0, \quad z(0) = z^0, \quad z^{\cdot}(0) = w^0 \tag{5.6}$$

to the given final state

$$y(T) = y^1, \quad y^{\cdot}(T) = v^1, \quad z(T) = z^1, \quad z^{\cdot}(T) = w^1 \tag{5.7}$$

The quantities on the right-hand sides of Eqs.(5.6) and (5.7) are constants, and $T > 0$ is the as yet unknown time when the process ends.

The phase vector of system (5.5) is formed by the variables $y, y^{\cdot}, z$ and $z^{\cdot}$. Following the general scheme of Par.2 for constructing the control, we find the fundamental matrix given by conditions (1.5), and then its inverse

$$\Phi^{-1}(t) = \begin{Vmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & 1 & -t \\ 0 & 0 & 0 & 1 \end{Vmatrix} \tag{5.8}$$

The matrix $Q(t)$ of (2.1) is here the four-dimensional column vector

$$Q^T(t) = (-\sin t, \cos t, -t, 1) \tag{5.9}$$

while Eq.(2.1) takes the form

$$u(t) = -c_1 \sin t + c_2 \cos t - c_3 t + c_4 \tag{5.10}$$

The expression for the matrix $R(T)$, given by the second of Eqs.(2.2) and (5.9), and hence the solution of system (2.2), is greatly simplified if we put $T = 2\pi k$, $k = 1, 2, \ldots$. The matrix $R(T)$ then becomes

$$R(T) = \begin{Vmatrix} {}^1\!/_2 T & 0 & -T & 0 \\ 0 & {}^1\!/_2 T & 0 & 0 \\ -T & 0 & {}^1\!/_3 T^3 & -{}^1\!/_2 T^2 \\ 0 & 0 & -{}^1\!/_2 T^2 & T \end{Vmatrix}$$

We solve system (2.2) in the light of this expression for $R(T)$ and express the components of the vector x^* in terms of the boundary conditions (5.6), (5.7) with the aid of relations (1.7) and (5.8):

$$x_1^* = y^1 - y^0, \quad x_2^* = v^1 - v^0$$
$$x_3^* = z^1 - Tw^1 - z^0, \quad x_4^* = w^1 - w^0 \quad (T = 2\pi k)$$

We obtain

$$\begin{aligned} c_1 &= 2\,[T^2 (y^1 - y^0) + 12 (z^1 - z^0) - 6T (w^0 + w^1)]S^{-1} \\ c_2 &= 2\,(v^1 - v^0)T^{-1} \\ c_3 &= 6\,[4 (y^1 - y^0) + 2 (z^1 - z^0) - T (w^0 + w^1)]S^{-1} \\ c_4 &= 2\,[6T (y^1 - y^0) + 3T (z^1 - z^0) - (T^2 + 12)w^1 - \\ &\quad 2 (T^2 - 6)w^0]S^{-1} \quad (S = T (T^2 - 24)) \end{aligned} \tag{5.11}$$

It remains to choose the integer k in the relation $T = 2\pi k$ in such a way that the control (5.10), (5.11) satisfies the condition $|u| \leqslant 1$ for $t \in [0, T]$. By (5.10), (5.11), we have

$$\begin{aligned} |u(t)| &\leqslant |c_1| + |c_2| + |c_4 - c_3 t| \leqslant 2T^{-1} (T^2 - 24)^{-1} [T^2 \times \\ &|y^1 - y^0| + 12 |z^1 - z^0| + 6T |w^1 + w^0| + (T^2 - \\ &24)|v^1 - v^0| + 6 |y^1 - y^0|\,|T - 2t| + 3 |z^1 - z^0|\,|T - \\ &2t| + \psi(t)] \\ \psi(t) &= |(T^2 + 12)w^1 + 2 (T^2 - 6)w^0 - 3Tt (w^1 + w^0)| \end{aligned} \tag{5.12}$$

Here, $T = 2\pi k$, $k \leqslant 1$, so that $T^2 > 24$.

Since $\psi(t)$ has its maximum at one end of the interval $[0, T]$, we have

$$\begin{aligned} \psi(t) &\leqslant \max\{\psi(0), \psi(T)\} = {}^1\!/_2 \max\{|3T^2 (w^0 + w^1) - \\ &(T^2 - 24)(w^1 - w^0)|, \ |3T^2 (w^0 + w^1) + (T^2 - 24)(w^1 - \\ &w^0)|\} = {}^3\!/_2 T^2\, |w^0 + w^1| + {}^1\!/_2 (T^2 - 24)|w^1 - w^0| \end{aligned}$$

Note also that $|T - 2t| \leqslant T$, $t \in [0, T]$.

Using these estimates, we obtain from (5.12):

$$|u(t)| \leqslant T^{-1}[f_1(T)|y^1 - y^0| + 2|v^1 - v^0| + f_2(T)|w^1 + w^0| + |w^1 - w^0|] + 2T^{-2}f_2(T)|z^1 - z^0| \tag{5.13}$$
$$f_1(T) = \frac{2T^2+12T}{T^2-24}, \quad f_2(T) = \frac{3T^2+12T}{T^2-24}$$

On the right-hand side of (5.13) we replace the functions $f_1(T)$, $f_2(T)$, which are strictly decreasing for $T \geqslant T_1 = 2\pi$, by their maximum values at $T = T_1$, and in the resulting inequality we put $T = 2\pi k$, $T_1 = 2\pi$. We obtain

$$|u(t)| \leqslant Ak^{-1} + Bk^{-2} \tag{5.14}$$
$$A = \frac{\pi+3}{\pi^2-6}|y^1 - y^0| + \frac{1}{\pi}|v^1 - v^0| + \frac{3(\pi+2)}{2(\pi^2-6)}|w^1 + w^0| + \frac{1}{2\pi}|w^1 - w^0|, \quad B = \frac{3(\pi+2)}{2\pi(\pi^2-6)}|z^1 - z^0|$$

It follows from (5.14) that the condition $|u| \leqslant 1$ holds if

$$k^2 - Ak - B \geqslant 0,$$

i.e., if

$$T = 2\pi k, \quad k \geqslant k^* = {}^1/_2[A + (A^2 + 4B)^{1/2}] \tag{5.15}$$

Relations (5.10) and (5.11) together with (5.15) for T and (5.14) for A, B, fully define the required control $u(t)$ in explicit form in terms of the initial and final states.

We consider a special case of boundary conditions (5.6) and (5.7):

$$y^0 = v^0 = z^0 = w^0 = y^1 = v^1 = w^1 = 0 \tag{5.16}$$

corresponding to displacement of the entire system of Figs.1, 2, from the equilibrium state to the equilibrium state at a distance z^1. In the case (5.16), the time-optimal control is of the relay type ($u = \pm 1$) and has three switching points /5/. The optimal time T^0 is the unique positive root of the equation

$${}^1/_4(T^0)^2 - 2\{\arccos[\cos^2({}^1/_4T^0)]\}^2 = |z^1|,$$

while we have the relations

$$T^0 \geqslant 2|z^1|^{1/2}, \quad T^0 \sim 2|z^1|^{1/2} \text{ as } |z^1| \to \infty \tag{5.17}$$

Let us compare this result with the displacement time for the control law (5.10). From (5.14)-(5.16), we have

$$T = 2\pi(\text{ent } k^* + 1), \ k^* = B^{1/2} = 0,\ 7965|z^1|^{1/2}$$

Hence, for large $|z^1|$, we obtain

$$T \sim 5.005|z^1|^{1/2}, \quad |z^1| \to \infty \tag{5.18}$$

If we use the estimate **(5.13) directly** in the case (5.16) as $|z^1| \to \infty$, we obtain

$$T \sim [2f_2(\infty)|z^1|]^{1/2} = 6^{1/2}|z^1|^{1/2} = 2,449|z^1|^{1/2}, \quad |z^1| \to \infty \tag{5.19}$$

Comparing (5.17)-(5.19) for T^0, T, we see that, as $|z^1| \to \infty$, they differ by coefficients which are due, both to the difference of the control (5.10) from optimal, and to the majorization which is performed when obtaining estimate (5.14). Notice that estimate (5.19) is much closer to (5.17) than is (5.18), because of the reduced "loss" with majorization.

Notice in conclusion that the time-optimal controls for the problems considered in Paragraphs 3-5 are not known for arbitrary initial conditions.

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